# AN APPLICATION OF HABERMAS CONSTRUCT OF RATIONALITY TO SUPPORT STUDENTS' PROOF VALIDATION SKILLS

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*Proof validation plays a significant role in students' understanding and learning of mathematical proofs. Recent studies have shown that university students were lacking skills in proof validation and were challenged by the implementation of the appropriate acceptance criteria when validating proofs. Drawing on Habermas' construct and rational questioning, the present study develops a proof validation framework that seeks to improve students' proof validation skills. The written responses of students' proof validation were analyzed based on their use of the proof validation framework in the context of a transition-to-proof course. The results showed that students primarily focused on the epistemic rationality component when they accept or reject the purported proof and that the students experienced difficulties with meeting the requirements of rationality components. Some educational implications are provided.*

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Mathematical proof is an essential component of undergraduate mathematics courses and serves as a prerequisite for students' advanced mathematical learning. Research in mathematics education indicates that most students have experienced substantial difficulty with proofs at the undergraduate level or beyond (Selden, 2012; Sommerhoff & Ufer, 2019; Weber, 2010). Many universities have instituted *transition-to-proof courses* (cf. Moore, 1994) to help students to review methods of mathematical proof (e.g., direct proof, proof by contradiction, proof by induction) to equip them for advanced mathematics courses that require mathematical proving skills. Students are expected to be able to read, evaluate, and write proofs through such a transition-to-proof course. The research studies on these courses have primarily concentrated on investigating students' beliefs regarding proofs, as well as their approaches to and challenges with proof comprehension, construction, and validation (e.g., Ko & Knuth, 2013; Mejia-Ramos et al., 2012; Segal, 2000; Selden & Selden, 2015; Weber, 2010). A few studies on proof validation have reported that university students often encounter substantial obstacles in accurately validating proofs and offering solid justifications for their judgments (see Alcock & Weber, 2005; Bleiler et al., 2014; Ko & Knuth, 2013; Kirsten & Greefrath, 2023; Selden & Selden, 2003, 2015; Sommerhoff & Ufer, 2019). However, little empirical research exists on how to improve students' skills of *validation of proofs* as in Selden and Selden, 1995 which focused on "proofs as texts that establish the truth of theorems and on readings of, and reflections on, proofs to determine their correctness" (Selden & Selden, 2003, p. 5). Proof validation plays a significant role at university level mathematics because university students who take advanced mathematics courses are expected to spend substantial study time reading and evaluating proofs and arguments that are presented by lectures or textbooks (Inglis & Alcock, 2012; Selden & Selden, 2003; Weber, 2004). Researchers (e.g., Alcock & Weber, 2005; Kirsten & Greefrath, 2023; Selden & Selden, 2003; Sommerhoff & Ufer, 2019) have suggested that validation of proofs should be taught explicitly in the university.

Studies that have focused on proof validation have typically explored the ability of students to differentiate between valid and invalid arguments (e.g., Alcock & Weber, 2005; Bleiler et al.,

2014; Ko & Knuth, 2013; Selden & Selden, 2015; Weber, 2010). In his study of 28 undergraduate mathematics majors who had completed a transition to proof course, Weber (2010) found that the students judged invalid proofs to be valid proofs 60% of the time (64 out of 106). Furthermore, researchers have sought to gain insights into the cognitive processes involved in students' proof validation process by employing assessment models to better understand how students make these judgments. For instance, Selden and Selden (2003) found that undergraduates focused too much on *the surface features* of an argument, in which students tend to check proofs step by step rather than by the logical structure of proofs. Harel and Sowder (2007) proposed the idea of using "proof schemes" to describe and evaluate student performance in proving. The results of these studies revealed that many students judged invalid empirical arguments to be valid proofs because their ability to determine whether arguments were proofs was very limited, and they had problems with implementing appropriate acceptance criteria for validating proofs. However, these assessment models or proof schemes mainly focus on documentation of students' current situations or difficulties they experience while validating proofs, so these models cannot address the distance between the students' proof validation performance and the teacher's expectations. Consequently, several researchers (e.g., Selden & Selden, 2013, 2015; Sommerhoff & Ufer, 2019; Weber & Alcock, 2005) have advocated for further research aimed at developing instructional tools to address students' challenges with proof validation. They have argued that instruction in transition-to-proof courses should incorporate the introduction of these tools with the goal of enhancing students' proof validation skills.

Proof by contradiction (PBC) is an essential proof method across all mathematical content areas and is often viewed as more difficult than direct proof (Quarfoot & Robin, 2022). According to Robin and Quarfoot (2022), students need opportunities for validation of proofs using the PBC method before being able to link these experiences in their written work. The present study examined how a cohort of mathematics students used a proof validation framework that was adapted from Habermas' (1998) construct of rationality in the context of a transition-to-proof course to validate proofs that focused on PBC.

## Theoretical Framework

In the field of mathematics education, Boero (2006) started to use Habermas' (1998) construct of rationality as a theoretical framework to study various mathematical discursive activities (e.g., proving, argumentation, problem-solving) based on the three components of rationality: epistemic (inherent in the control of validation of statements), teleological (inherent in the strategic choice of tools to achieve the goal of the activity), and communicative (inherent in the conscious choice of suitable means to communicate understandably within a given community). In recent years, following Boero (2006), several researchers (e.g., Boero et al., 2010; Boero & Planas, 2014; Morselli & Boero, 2011; Urhan & Bülbül, 2022a, 2022b; Zhuang, 2020; Zhuang & Conner, 2018, 2020, 2022a) have used Habermas' construct of rationality to analyze students' behavior in proving and problem-solving processes and developed instructional tools to help teachers plan and manage argumentative discourse according to rationality components. Moreover, considering the gap exists between the rationality of teachers and students, some researchers (e.g., Boero & Planas, 2014; Boero et al., 2018; Urhan & Bülbül, 2022; Zhuang & Conner, 2022a) have suggested the explicit introduction of Habermas' construct of rationality to students for pedagogical purposes to facilitate students' awareness of rationality requirements in proving and argumentation activities according to the teacher’s expectations.

Drawing on Habermas' (1998) construct of rational behavior, Zhuang and Conner (2018, 2022a) developed a *rational questioning framework* for teachers to support students in meeting the requirements of rationality in argumentation. The present study views rational questioning as a didactical tool that can be introduced to students in order to guide them in the validation of proofs as shaped by rationality requirements; because rational questioning guide students to concern the choice of appropriate and efficient methods (teleological rationality) on the basis of mathematical knowledge (epistemic rationality), such as rules, theorems, axioms, and principles and communicate in a precise way with the choice of understandable means of communication within the shared mathematical community (communicative rationality), which corresponds to what policy documents and mathematics educators suggest as important characteristics of proof in mathematics (Boero et al., 2010; Boero & Planas, 2014; NCTM, 2000; Stylianides, 2007).

By adapting Habermas' (1998) construct of rationality and Zhuang and Conner's (2018, 2022a) rational questioning framework, this study develops a *proof validation framework* (see Table 1) to support students' proof validation skills via the use of rational questioning. The purpose of this study is to explore how students validate the p*urported proofs* (cf. Selden & Selden, 2003) through the use of the developed proof validation framework. More specifically, this study addresses the following research questions:

1. Which rationality component is privileged when students accept or reject the purported proof?
2. Which rationality components students are competent when they validate the purported proof?

**Table 1: Proof Validation Framework**

| Components of Habermas' Rationality | Rational questioning to consider when validating purported proofs |
|---|---|
| Epistemic Rationality (*ER*):<br>Uses valid definitions, axioms, and theorems shared by the mathematical community. | E1. What are the mathematical definitions, axioms, or theorems stated in the proof? Are they true?<br>E2. Are there any other definitions, axioms, or theorems that would more reliably account for the stated mathematical claims?<br>E3. What are the warrants or reasons used to support the stated mathematical claims and mathematical arguments?<br>E4. Does the proof provide correct warrants or reasons to justify the stated mathematical claims and mathematical arguments?<br>E5. Are the warrants or reasons convincing enough to help someone understand why the stated mathematical claims are true? |
| Teleological Rationality (*TR*):<br>Employs efficient proof strategies (e.g., direct proofs, proof by | T1. What the proof methods (e.g., proof by contradiction, direct proof) are used to prove the stated mathematical claims and |

| | |
|---|---|
| contradiction, proof by induction, etc.) to achieve the goal of proof. | mathematical arguments? Are the proof methods used correctly and logically to achieve the goal of the proof?<br>T2. How efficient is the applied proof method? Could any other methods be taken into account?<br>T3. Could these proof methods be used to solve a similar problem? |
| Communicative Rationality (*CR*): Writes with forms of expression (e.g., mathematical language, visual representations, symbolic notation, etc.) that are understandable in the mathematical community. | C1. Are the proofs or arguments represented clearly?<br>C2. Do the proofs or arguments contain mathematically correct language, visual representations, or symbolic notations?<br>C3. Are there any ways to organize or write the proofs or arguments more clearly? Are there any irrelevant or distracting points? |

## Methodology

By following an instrumental case study (Stake, 1995), this study analyzed a cohort of mathematics graduate students' written responses from one of their course assignments in an online transition-to-proof course. The course serves as an introductory course for a master's mathematics program at a comprehensive state university in the United States to help students to review methods of mathematical proofs before they take advanced graduate-level mathematics courses. Because it is an online course, the participants of this study come from all over the world. Most participating students hold a bachelor’s degree related to mathematics (i.e., pure mathematics, applied mathematics, mathematics education) or have studied a STEM-related major (e.g., computer science) for their undergraduate degrees. A few students have some experience in proof-related mathematics courses, such as advanced calculus and abstract algebra, but none of them have taken a transition-to-proof course that specifically focuses on proof validation and written techniques. Many students in this class hope to teach mathematics at a community college after completing the master's program in mathematics.

During the first couple of weeks of the course, the instructor introduced the proof validation framework (see Table 1) to the class and guided students to use the framework to validate the proofs that focused on direct proof methods through an online interactive learning community. Next, the students were expected to use the proof validation framework to complete the weekly Proof Writer's Workshop (PWW) assignment in which the students needed to read, evaluate, and critique at least one purported proof that is common to the new proof methods that are covered in the course. Students had to decide whether the purported proof that was given by the instructor should receive a pass (solid) proof, a revised (developing proof), or a failed grade (flawed proof), and the students were asked to justify their validations. In the assignment, the students were given a reflection question about the use of the proof validation framework: "Which components of rationality help you analyze this proof/argument? Explain." after they evaluated the purported proof.

The data in this study include 16 participating students' written responses on a purported proof that focused on PBC. The purported proof and the reflection question from their PWW

assignments are shown in Figure 1. Before working on the PWW assignment, students were assigned to read the textbook *How To Prove It* (Velleman, 2019) in regard to PBC content. For this purported proof, the instructor expects the students to determine whether a fundamental flaw exists in reasoning about the definition of set difference (epistemic rationality). Although the PBC was used correctly, it may not be the most efficient proof method to apply, proof by contrapositive can be considered as a more efficient strategy (teleological rationality). This purported proof is well-written in text, but it could be more consistent with symbolic notations, such as the use of the set difference signs as "\" or "-" (communicative rationality).

This study conducted a microanalysis of each student's written responses for one of their weekly PWW assignment (Corbin & Strauss, 2015). The responses provided by the students for the first question were utilized to assess the accuracy of their proof validation, as well as to analyze the rationality components in which students demonstrated competence during the validation of the proposed proof (*RQ2*). Through the use of the constant comparative method (Glaser & Strauss, 1967), the students' answers to the reflection question were used to investigate which rationality component is privileged when they make judgments (*RQ1*).

The goal of the proof writer's workshop assignment is to help you get a better sense of what counts as a "solid proof", a "developing proof" and a "flawed proof". Each assignment will have at least one proof/argument for you to read, evaluate and critique. You will take the role of the grader: your job is to decide whether each proof/argument should receive a pass (solid proof), a revise (developing proof) or a failed grade (flawed proof).

**Theorem 1:** Suppose $A \subseteq B$ and $C$ is any set. If $x \in A - C$ then $x \in B - C$.

*Proof.* We will give a proof by contradiction. Suppose for contradiction that $x \in A \setminus C$ but $x \notin B \setminus C$. From the definition of set difference, this tells us that $x \notin B$ and $x \in C$. But $x \in A \setminus C$ tells us that $x \in A$ and $x \notin C$. It is impossible to have $x \in C$ and $x \notin C$, so we have a contradiction. Therefore the theorem statement is true. □

**Q1**: Overall, would you rate this proof/argument as a pass or a revise? Or do you think there is a fundamental flaw in this proof/argument? Please justify your decision. You may consider evaluation proof/argument via the use of the Proof Validation Framework.

**Reflection Question**: Which component(s) of rationality help you analyze this proof/argument? Explain.

**Figure 1: Proof Writer's Workshop (PWW) Assignment Used in this Study**

## Results

Through the analysis of the students' PWW assignment written responses, 14 out of 16 students gave the purported proof (see Figure 1) either a revised or a failed grade. Interestingly, the two students who made the wrong evaluation, in which they gave a grade of pass, did not validate the purported proof based on rationality components. This section reports how these 14

students validated the purported proof through the use of the proof validation framework (see Table 1).

**Privileged Rationality Component**

Based on students' written responses to the reflection question, all 14 students stated that they evaluated the given purported proof through epistemic rationality. Most of the students drew on epistemic rationality to analyze the correctness of the definition of set difference (under the guidance of E1 rational questioning, see Table 1). For instance, one student responded, "The epistemic rational element was the primary element that helped me analyze this proof because not stating a valid definition is what leads to the flaw in reasoning. You could also say that a valid definition (set difference) was incorrectly applied." A few students pointed out that the purported proof had a lack of warrants or reasons to support the stated theorem because the definition of the subset (given $A \subseteq B$) should be employed (under the guidance of E4 rational questioning).

Eight students mentioned that they applied teleological rationality when they validated the purported proof. One student highlighted that "The main rationality element that helped the most with this proof is teleological because the use of PBC seemed odd to me." The students generally drew on teleological rationality to evaluate the correctness of the proof method (under the guidance of T2 rational questioning, see Table 1). However, students rarely focused on the effectiveness of the proof method (under the guidance of T3 and T4 rational questioning). Only one student suggested that the proof could be shorter if a direct proof method were employed.

Eight students concentered on the communicative rationality of the purported proof. Most of them were satisfied with the language and notation that was used in the given purported proof (under the guidance of C1 and C2 rational questioning). Some students have different opinions as to whether the proof method should be stated in the beginning. For instance, one student stated that "It was also nice to have them state the proof method in the first sentence, so the reader knows what assumptions should be expected." Another student commented, "Maybe a little redundant to say proving by contradiction and then stating again that they were supposing for contradiction."

Overall, when validating the given purported proof, 7 students articulated that all three components of rationality helped them to validate the purported proof. Five students merely mentioned epistemic rationality. One student focused on both epistemic and teleological components of rationality and another focused on both epistemic and communicative components of rationality. In this sense, epistemic rationality appears to be the privileged rationality component when students accept or reject the given purported proof.

**Competent Rationality Components**

Among students who focused on the epistemic rationality component, 11 students were able to determine the correctness of the definition of set difference as stated in the purported proof, "From the definition of set difference, this tells us that $x \notin B$ and $x \in C$. " In addition, five of them corrected the definition in their written responses either by stating the correct definition or providing a counterexample. On the other hand, three students thought that the writer stated the definition of set difference correctly.

Of eight students who attended the teleological rationality component, four thought that the PBC method was applied correctly. Another four students articulated that the writer did not follow the PBC strategy due to incorrect assumptions. For instance, one student stated that "The use of contradiction is wrong since the contradiction used, $x \notin B - C$ which contradicts the

primary given assumption." The rest of the three students argued that the writer should assume $x \notin A - C$ because according to their understanding of PBC, the writer should assume the negation of the *if* portion of the statement.

The students who have paid attention to the communicative rationality all agreed that the purported proof was written clearly in the text. As one student commented, "The proof was written clearly and used correct notations." Several students recommended improving the written work by ensuring consistency in the usage of notations, such as the set difference sign.

The results of this study also showed that 3 out of 14 students provided both invalid interpretations of epistemic and teleological rationality even though they rated the purported proof as revised. In addition, the two students who gave the grade of pass did not apply the proof validation framework. Without focusing on any rationality component, one student simply commented that "This proof looks to be sufficient to support the conclusion." Another student applied the direct proof method to show that the given purported proof was true in terms of the theorem and therefore concluded that the presented purported proof should pass with a lack of concentration on the purported proof itself.

## Discussion

Researchers have used Habermas' (1998) construct to identify and interpret the challenges that are experienced by university students in mathematical proving and problem-solving activities (e.g., Boero & Morselli, 2009; Urhan & Bülbül, 2022b). This study responds to the call within the field to conduct research that develops approaches to teaching proof validations explicitly. The results of this study indicate that the proof validation framework (see Table 1) based on Habermas' (1998) construct of rationality and guided by rational questioning (Zhuang, 2020; Zhuang & Conner, 2018, 2022a) provides students with a tool to validate mathematical proofs in terms of rationality components rather than simply focused on what Selden and Selden (2003) called the surface features of proofs and arguments. According to Boero et al. (2010), the goal of developing rational behavior in proving must be guided and promoted by teachers. The introduction of the proof validation framework to students scaffolds them to be aware of the rationality requirements inherent in proving and to facilitate students' maturation of acting rationality in proving activities in a long-term teaching intervention. As one student replied in the reflection question, "All three of the elements of rationality helped me to analyze this proof. I found that the rational questions provided helped me the most. The questions allowed me to look for specific reasoning in the proof. This allowed me to break apart the proof, sentence by sentence, and apply the rational elements."

Previous research that examined students' abilities in proof validation has primarily concentrated on assessing the accuracy of their judgments regarding the warrants or reasons presented in the purported proofs (Alcock & Weber, 2005; Selden & Selden, 2003; Sommerhoff & Ufer, 2019; Weber & Alcock, 2005). Habermas' construct provides us with a more comprehensive frame in which to understand varying aspects of students' acceptance criteria for validating mathematical proofs, especially for teleological rationality and communicative rationality. The construct enables teachers to identify students' competence in rationality components and embeds students' competence in rationality components into their classroom instruction.

The result of this study showed that the epistemic rationality was represented as the privileged rationality component when students validated the given purported proof. This finding is unsurprising given that the principal flaw inherent in the purported proof lies in the incorrect

definition of the stated set difference. This finding is also consistent with previous studies (see Alcock & Weber, 2005; Inglis & Alcock, 2012; Ko & Knuth, 2013; Kirsten & Greefrath, 2023; Selden & Selden, 2003; Sommerhoff & Ufer, 2019) in which students tended to check proofs line by line to make sure that each statement in the argument is mathematically correct. In addition, this study found that only 8 out of 14 students considered either the teleological or the communicative rationality component when they made judgments. This result implies that students prioritized the evaluation of the truthfulness of the stated assertions, rather than critically assessing the appropriateness and effectiveness of the employed proof method, as well as the clarity of the proof presentation. What is most disappointing is that only a single student acknowledged the efficiency of utilizing PBC in the purported proof. Hence, it is imperative for teachers to provide guidance to students regarding the teleological and communicative rationality during the process of proof validation, in order to direct the students' attention toward the overall strategy employed and the written structure of the proof.

Although the students who applied the proof validation framework (see Table 1) correctly rated the purported proof as either revised or failed under the guidance of rational questioning, it was noticed that some students' interpretations of rationality components were incompetent. The failures of interpretations include inadequate knowledge of the definition of set difference (epistemic rationality) and the incorrect assumption of negating *if* statement regards PBC (teleological rationality). These results demonstrate the importance to ask students to provide justifications for their validation according to the three rationality components because it enables teachers to determine the rationality components at which the students are competent. In this way, teachers can plan and design the follow-up process of teaching not only by focusing on students' gaps in mathematical knowledge but also by taking the rationality components into account based on the rationality requirements that teachers expect students to achieve in proving activities.

## Implications and Future Directions

A number of studies (e.g., Kirsten & Greefrath, 2023; Selden & Selden, 2003, 2015; Sommerhoff & Ufer, 2019) have consistently revealed the challenges encountered by university students in the process of validating proofs. However, the skills critical for proof validation often do not receive adequate emphasis within the mathematics classroom.

This study proposes the idea of introducing the proof validation framework (see Table 1) to students in a transition-to-proof course. This framework provides a way for students to implement rationality components when they validate proofs, so they may benefit from interventions that focus on the requirements of rationality behavior in proving activities. Moreover, the utilization of the proof validation framework enables teachers to evaluate students' proof validation performance at a meta-level, as described by Boero et al. (2010), encompassing an awareness of the constraints associated with the three components of rationality as well as the content of the proof. For pedagogical purposes, the proof validation framework scaffolds students to realize proving as a rational process and guides them to gradually move to an awareness of epistemic, teleological, and communicative requirements of rationality that is inherent in proving as a long-term teaching intervention.

The present study proposes one potential way to improve students' proof validation skills, and other possible ways to improve students' skills in proof validation are worthy of investigation. Zhuang and Conner (2022b) discussed the use of students' incorrect answers through classroom-based argumentation, so it may be beneficial to examine errors in proofs that

are constructed by students. Additionally, it would be interesting to investigate whether students are able to apply the skills of proof validation in terms of rationality components to their construction of proofs.

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